Assignment 5

This homework is due Friday Feb 27.

There are total 45 points in this assignment. 40 points is considered 100%. If you go over 40 points, you will get over 100% for this homework (but not over 115%) and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper *and give credit to your collaborators in your pledge.* Your solutions should exhibit your work and contain full proofs. Bare answers will not earn you much.

This assignment covers Sections 2.5–3.2 of Textbook.

(1) [10pt] Find the images of the mapping $w = 1/z$ in each case, and sketch the mapping.
 (a) The horizontal line $\{(x, y) : y = \frac{1}{4}\}$.
 (b) The vertical line $\mathrm{Re}(z) = -3$.
 (c) The circle $C_{\frac{1}{2}}(-\frac{i}{2}) = \{z : |z + \frac{i}{2}| = \frac{1}{2}\}$.
 (d) The circle $C_1(-2) = \{z : |z + 2| = 1\}$.
 (e) The line $2x + 2y = 1$.

(2) [5pt]
 (a) Show that transformation $w = 1/z$ maps the vertical strip given by $0 < x < \frac{1}{2}$ onto the region in the right half-plane $\mathrm{Re}(w) > 0$ that lies outside the disk $D_1(1) = \{w : |w - 1| = 1\}$.
 (b) Find the image of the disk $D_{\frac{4}{3}}(\frac{-2i}{3}) = \{z : |z + \frac{2i}{3}| < \frac{4}{3}\}$ under $f(z) = 1/z$.

(3) [5pt] Prove the following directly by computing the limit in the definition of the derivative.
 (a) $(z^3)' = 3z^2$.
 (b) $\left(\frac{1}{z}\right)' = \frac{-1}{z^2}$.

(4) [5pt] Find the derivative of the following functions using rules of differentiation.
 (a) $(z^2 - iz + 9)^5$. (Simplifying the answer is not necessary.)
 (b) $\frac{2z+1}{z+2}$.
 (c) $(z^2 + (1 - 2i)z + 1)(z^2 + 3z^2 + 5i)$.

(5) [10pt] Use the Cauchy–Riemann conditions to determine where the following functions are differentiable and evaluate the derivatives at those points where they do exist.
 (a) $f(z) = f(x, y) = \frac{y+ix}{x^2+y^2}$.
 (b) $f(z) = -2(xy + x) + i(x^2 - 2y - y^2)$.
 (c) $f(z) = x^3 + i(1 - y^3)$.
 (d) $f(z) = x^3 - 3x^2 - 3xy^2 + 3y^2 + i(3x^2y - 6xy - y^3)$.
 (e) $f(z) = x^2 + y^2 + 2ixy$.

(6) [5pt]
 (a) Use any method to show that the function $h(z) = e^y \cos x + ie^y \sin x$ is not differentiable anywhere.
 (b) Show that the function $g(z) = \cosh x \sin y - i \sinh x \cos y$ is entire.

(7) [5pt] Let f and g be analytic functions in the domain D. If $f'(z) = g'(z)$ for all z in D, then show that $f(z) = g(z) + C$, where C is a complex constant. (*Hint:* Consider $f - g$. Or, look at Re and Im separately and use "usual" multivariable calculus.)